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THE THEORY OF IMAGES IN THE REPRESENTATION OF FUNCTIONS.*

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1. If u be a function of z, and if z describe any path, u will describe another path which is called the *image* of the former with reference to the given function.

The problem constantly before us in the following discussion, is that of ascertaining points in the u-plane which correspond to given points in the z-plane (thereby *representing* the function), without involving needless computation.

2. The *transforming equation* is the relation subsisting between u and z. It will usually be indicated thus:

$$\Phi\,(u, z) = 0, \quad \text{or} \quad u = \varphi\,(z).$$

Either Φ or φ may be called the *transforming function.*

3. The *path*, when not otherwise specified, is the course of the independent variable. Its equation will usually be written

$$f(x, y) = 0.$$

We shall indicate the equation of the image, with respect to Φ, thus:

$$\mathbf{I}^{f(x,y)}_{\Phi(u,z)} = 0, \quad \text{or} \quad \mathbf{I}^{f}_{\Phi} = 0.$$

4. If u be a many-valued function of z, m denoting the order of multiplicity; and if z be a many-valued function of u, n denoting the order of multiplicity; then to each value of z there correspond, in general, m distinct values of u, and to each value of u correspond n of z.

In this case either the path, or the image, or both, *may* be *multiform.*

5. By supposing f to contain a variable parameter, we obtain a *system of paths.* The corresponding courses of u constitute an *image of the system.*

When for a particular value of the parameter, the degree of the equation of the image is lowered in respect to either variable or to both variables (the curve then breaking up into curves of lower degree), this line will be called a *path of reduction*, and the corresponding course of u, a *reduced image*.

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EXAMPLES OF PATHS OF REDUCTION (Φ REAL).

(1.) The axis of x for the system $y = \beta$. For, if we equate the imaginary part of $\Phi(u, z)$ to zero, every term must contain a y or Y. Hence when $y = 0$, Y will be a factor of the equation just formed.

(2.) If the order of each of the terms of Φ be odd (and the constant term wanting) then the axis of y is a line of reduction in the system $x = a$.

(3.) If the variables u, z be separate, if u enter by odd powers only, and if the system be the real part of the function of z equated to k, that value of k which cancels the constant in Φ will give a curve of reduction.

(4.) On the other hand, if u and z be separate and the purely imaginary part of the function of z equated to k', $k' = 0$ will always give a curve of reduction.

(5.) Every quardratic function with real coefficients will be shown to have a line of reduction parallel to the y-axis. Art. 21.

6. *Translation* is the result of substituting for u, $u' - \nu$, and for z, $z' - \nu'$, in the transforming equation, and simultaneously for x, $x' - \nu_1'$, and for y, $y' - \nu_2'$, in the equation of the path. ν, ν' are any constants; $\nu_1' + i\nu_2' = \nu'$. That is, if we move the origin $O_{z'}$ to the point whose affix is ν', any path of z' becomes a path of z referred to O_z. If we move $O_{u'}$ to the point ν, then any course of u' becomes a course of u referred to O_u.

7. *Imaginaries* is a generic term applied to all quantities not real; these latter may be either *complex quantities*, or *pure imaginaries*.

8. For the sake of brevity, we shall denote the real part of any expression or function F, by $\mathfrak{r}F$, and the purely imaginary part by $i\mathfrak{p}F$.

9. Expressions whose terms are homogeneous in each of two sets of letters, as $X^m y^n$, $X^{m-r} Y^r x^{n-s} y^s$, etc., may be denoted thus:

$$(X, Y)^m (x, y)^n.$$

10. *If z describe any infinitesimal element, u describes a similar one.*

Let the arguments of dz and du be ω and χ, respectively; σ, ρ their moduli.*

Then $$dz = \sigma e^{i\omega}, \quad du = \rho \varepsilon^{i\chi};$$

$$\therefore \frac{du}{dz} = \frac{\rho}{\sigma} e^{i(\chi - \omega)}. \tag{1}$$

But $$\frac{du}{dz} = \frac{\partial X}{\partial x} + i\frac{\partial Y}{\partial x} = \frac{\partial Y}{\partial y} - i\frac{\partial X}{\partial y}. \tag{2}$$

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*Hoüel, *Calcul Infinitésimal,* Vol. 3. p. 249.

From (1) and (2) we get

$$\frac{\rho}{\sigma}=\sqrt{\left[\left(\frac{\partial X}{\partial x}\right)^2+\left(\frac{\partial Y}{\partial x}\right)^2\right]},\quad \chi-\omega=\tan^{-1}\frac{\frac{\partial Y}{\partial x}}{\frac{\partial X}{\partial x}}$$

Hence $\chi-\omega$ depends only upon the position of z, and is always the same for that point regardless of the direction in which dz is taken. If, then, we increase or diminish the argument of dz by any given amount, the argument of du will be increased or diminished by an equal amount. Hence all angles of the infinitesimal elements, described by z, are preserved in the images of these paths.

If $u=\varphi(z)$, and if $v=\psi(u,z)$, then of course, the elements described by v are similar to those described by u or z.

11. By taking the inverse function along the image, we get the original curve or path as an image of the image.

The truth of this proposition is obvious. Since any point z, operated upon by φ gives u, or $\varphi(z)$, if we operate upon this result by φ^{-1}, the two operators mutually destroy each other.

12. If we develop $u=\varphi(z)$ by Taylor's theorem, we shall have

$$u=\varphi(x)+\frac{iy}{1}\frac{\partial\varphi(x)}{\partial x}+\frac{(iy)^2}{2!}\frac{\partial^2\varphi(x)}{\partial x^2}+\ldots+\frac{(iy)^n}{n!}\cdot\frac{\partial^n\varphi(x)}{\partial x^n};$$

$$\therefore X=\varphi(x)-\frac{y^2}{2!}\frac{\partial^2\varphi(x)}{\partial x^2}+\frac{y^4}{4!}\frac{\partial^4\varphi(x)}{\partial x^4}-\ldots\equiv\cos\left(y\frac{\partial}{\partial x}\right)\cdot\varphi(x),\qquad(1)$$

$$Y=y\frac{\partial\varphi(x)}{\partial x}-\frac{y^3}{3!}\frac{\partial^3\varphi(x)}{\partial x^3}+\frac{y^5}{5!}\frac{\partial^5\varphi(x)}{\partial x^5}-\ldots\equiv\sin\left(y\frac{\partial}{\partial x}\right)\cdot\varphi(x).\qquad(2)$$

If we eliminate x and y from (1), (2), and $f(x,y)=0$, the resultant equated to zero will be the equation of the image.

SYMMETRY; φ REAL.

13. *If $f(x,y)=0$ be symmetric about the x-axis, then is the image symmetric about the X-axis.*

This is equivalent to the statement that if

$$\varphi(x+iy)=X+iY,$$

then

$$\varphi(x-iy)=X-iY.$$

When the path is symmetric about the y-axis, then if φ be an $\overset{\text{odd}}{\text{even}}$ function, I is symmetric about the $\overset{\text{Y-axis}}{\text{X axis}}$.

THE ORDER OF I, WHEN f AND Φ ARE RATIONAL ALGEBRAIC FUNCTIONS.

14. *Its limit.* Let m denote the highest power in which u enters Φ, n the highest power of z, and let p be the order of f.

Then the (x,y)-resultant of

$$\mathfrak{r}\Phi = 0, \quad \mathfrak{p}\Phi = 0, \quad f = 0,$$

i. e., the resultant got by eliminating (x,y), is of the pnth degree in the coefficients of each of the first two equations. But these coefficients contain X, Y to an order not greater than m; hence the order of I is not greater than $2mnp$.

15. *General method of ascertaining the order.*

Let
$$U_\alpha z^n + U_\beta z^{n-1}v + \ .\ .\ . + U_\tau v^n = 0, \tag{1}$$
represent the general transforming equation; let $f(x,y,v) = 0$ be the equation of the path. v is introduced for the sake of homogeneity and is always understood to be equal to unity. U_α, U_β, etc., are polynomials of the orders α, β, etc. in u. Expanding (1), and equating $\mathfrak{r}\Phi$ and $\mathfrak{p}\Phi$ separately to zero, we have

$$(\grave{}X, Y\underline{)^{\alpha}\ }(x,y)^n + (\grave{}X, Y\underline{)^{\beta}\ (}x,y)^{n-1}v + \ .\ .\ . + (\grave{}X, Y)^{\tau}\, v^n = 0,$$
$$(\grave{}X, Y\underline{)'^{\alpha}}(x,y)'^{n} + (\grave{}X, Y\underline{)'^{\beta}}(x,y)'^{n-1}v + \ .\ .\ . + (\grave{}X, Y)'^{\tau}\, v^n = 0.$$

The grave accent is used to indicate that the function is not in general homogeneous in X, Y.

Suppose that the coefficients of all terms of $\mathfrak{r}\Phi$, $\mathfrak{p}\Phi$ containing v to a given power be affected with a suffix equal to this power, then the (x,y)-resultant will be a homogeneous function in these coefficients of order $2np$ and weight n^2p. Let a_0 denote any coefficient of v^0 containing X, Y to the α power; a_1, any coefficient of v containing X, Y to the β power, etc. Form out of the a's all possible terms having the required order and weight. Observe the order of each a (regarded as a function of X, Y) that enters any given term as a factor. Let θ denote this order, and let t be the number of times that it enters as a factor; then

$$\Sigma(t\theta)$$

is the order of the particular term in question. The highest order found, when the successive terms are thus treated, is the order of I.

16. *To find the degree to which a_r may enter the resultant.*

Let us first consider the special case where $p = 1$, and let t denote the degree in question. Then, since no subscript can exceed n, we have (because weight $\div$ order $=$ subscript),

$$\frac{n^2 - rt}{2n - t} \leqq n; \quad \therefore t \leqq \frac{n^2}{n - r}.$$

If we use E to indicate that only the integral part of the quantity is taken, we have

$$t = E\left(\frac{n^2}{n-r}\right). \tag{1}$$

This formula holds true whenever $n \geqq 2r$. When $n < 2r$, we must use the obvious formula

$$t = E\left(\frac{n^2}{r}\right). \tag{2}$$

Reasoning as before, we have in general

$$t = E\left(\frac{n^2 p}{n-r}\right), \quad n \geqq 2r, \tag{3}$$

$$t = E\left(\frac{n^2 p}{r}\right), \quad n \leqq 2r. \tag{4}$$

If Φ be a homogeneous function in u and z of order n, then the image of any curve of the pth degree is of the order $n^2 p$. For the subscripts of the a's in this case denote their orders in X, Y. Hence the weight of any term will be the order of I. But we know this to be $n^2 p$. So, if $\alpha, \beta, \ldots \tau \leqq 0, 1, \ldots n$, respectively, even if Φ be not homogeneous, the degree of I is at most $n^2 p$.

Φ OF THE FIRST DEGREE.

17. When Φ is of the first degree in u and z, the *nature* of any path transformed will be preserved. In other words, the path will at most be translated, rotated, and altered in scale.

Let the transforming equation be

$$u = kz + c.$$

Take any point z_1 upon the curve $f = 0$. If we multiply it by the affix of k, we increase its argument by α, the argument of k, and multiply its modulus by l, the modulus of k. We then translate the resulting point a distance c. So, for every point on $f = 0$.

Let $z = ku$, and suppose the modulus of k to be equal to unity. Then,

$$x = X\cos\alpha - Y\sin\alpha, \tag{1}$$

$$y = X\sin\alpha + Y\cos\alpha, \tag{2}$$

or

$$z = e^{i\alpha}u. \tag{3}$$

Thus the well-known formulae (1) and (2) can be written in the form (3).

Φ OF THE SECOND DEGREE.

18. We shall begin the study of these functions by examining cases which

are special, either because certain coefficients are wanting, or because the coefficients are real, or for both reasons.

(1.) Suppose that $a, h = 0$ in the general equation

$$au^2 + bz^2 + c + 2fz + 2gu + 2huz = 0. \qquad (1)$$

By translation, this equation may now be written in the form

$$u' = b'z'^2. \qquad (2)$$

Consider the equation $$u'' = z'^2,$$

or $$R'' \operatorname{cis} \theta'' = \rho'^2 \operatorname{cis} 2\theta';$$

$$\therefore \; R'' = \rho'^2, \quad \theta'' = 2\theta'.$$

Now cause z' to describe a circle about O_z' with a radius ρ_1'; u'' will describe a circle about O_u' of radius $\rho_1'^2$. $\theta'' = 2\theta'$; hence, when u'' has made a complete revolution, z' has made but half of one. We have, then, for each value of u'' two of z' (which differ only in sign) in our representation. u' is obtained by multiplying the modulus of u'' by that of b', and increasing the argument of u'' by that of b'.

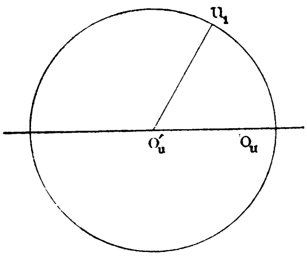
EXAMPLE. $$u + 2 = z^2 + 2z.$$

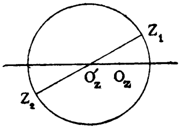
Here $u + 3 = (z + 1)^2$, $u'' = u' = u + 3$, $b' = 1$, $z' = z + 1$.

That is, if we measure the co-ordinates of z_1, z_2 referred to O_z, and of u_1 referred to O_u, the values thus obtained should satisfy the equation

$$u + 2 = z^2 + 2z.$$

In the case of real coefficients, the two systems of straight lines $x = \alpha, y = \beta$ are transformed into two systems of confocal parabolas, all axes of the curves of both systems lying along a line parallel to the axis of X.





(2.) Suppose $b, h = 0$; the equation becomes, by translation, of the form

$$u^2 = f'z.$$

Obviously, this may be represented by aid of circles, as was the equation of the form $u = b'z^2$ in the last article.

If z move along any straight line, u always describes an equilateral hyperbola.

For, let $$f' = f_1' + if_2',$$

then $$X^2 - Y^2 = f_1'x - f_2'y, \qquad (1)$$

$$2XY = f_2'x + f_1'y. \qquad (2)$$

Substituting for y in (1), (2) its value from the equation of the line, and eliminating x, we see that the axes of the hyperbola are equal.

In the case of real coefficients, the image of $x = \alpha$ is

$$\frac{X^2}{f'\alpha} - \frac{Y^2}{f'\alpha} = 1,$$

and that of $y = \beta$ is

$$2XY = f'\beta.$$

(3.) Suppose a, $b = 0$; the general equation now becomes, by translation, of the form

$$uz = c'.$$

Suppose $c' = 1$; then $u = \frac{1}{z}$, or

$$R \text{ cis } \Theta = \frac{1}{\rho} \text{ cis } (-\theta);$$

$$\therefore R = \frac{1}{\rho}, \quad \Theta = -\theta.$$

Hence, *the substitution* $u = \frac{1}{z}$ *gives the inverse of any curve along which* z *is taken at the same time inverting it.*

Multiply this locus by c' and we have the required curve.

(4.) If $h = 0$, the general equation may be written in the form

$$a'u^2 + b'z^2 = c'.$$

For the systems $x = \alpha, y = \beta$ the images are always of the form (if the coefficients are real)

$$Y = \pm k\sqrt{\left(\frac{X^2 \pm C^2}{X^2 \pm B^2}\right)},$$

where the ± signs under the radical depend upon the case considered.

This curve admits of a mechanical construction. We use for this purpose two right-angled triangles having a side in common whose length is X.

GENERAL CASE WITH REAL COEFFICIENTS.

19. Having by translation deprived the equation of terms of the first degree, we may write it in the form

$$au^2 + 2huz + bz^2 = c.$$

If, now, z be made to move along the line $x = \alpha$, the image becomes of the form

$$Y^2 = \frac{F_4(X)}{F_2(X)},$$

wherein F_4 and F_2 denote polynomials of the fourth and second degrees, respectively.

For $y = \beta$, I has the same form after interchanging X and Y.

Hence, *the images of all lines whose equations are $x = a$, or $y = \beta$ can be easily constructed when Φ is a real quadratic function.*

GENERAL CASE.

20. By linear substitution, we can put $\Phi_2(u, z) = 0$ into the form

$$u'z' = c.$$

Now, if z' be taken along any straight line, u' will describe a circle, as follows from 18 (3). Hence, this auxiliary image is easily constructed. Suppose such a system to have been thus constructed. Then u and z are obtained from it by the relations

$$u = \lambda u' + \mu z' + \nu,$$
$$z = \lambda' u' + \mu' z' + \nu'.$$

Since the courses of u are images of the courses of z with reference to the original function, we have a representation as required.

COROLLARY. z may now be taken along more convenient systems and, by referring to the u- and z-charts already constructed, the corresponding course of u be determined.

EXAMPLE. $$4u^2 + 6z^2 - 11uz + 14u - 13z + 256 = 0. \quad (1)$$

Here, $$u = 2u' + 3z' + 1, \quad (2)$$

$$z = u' + 4z' + 2; \quad (3)$$

$$\therefore u' = \tfrac{1}{5}(4u - 3z + 2), \quad z' = \tfrac{1}{5}(2z - u - 3); \quad (4)$$

$$u'z' = 10.$$

Let z' move over two systems of right lines whose equations are $x = a, y = \beta$. In the diagram, a varies from $\pm \frac{1}{2}$ to ± 4 and β has the same limits. When the paths of z' lie in the first angle, since $\theta' = -\theta'$, their images lie in the fourth angle, etc.

From these two charts we construct Figs. 3 and 4, by aid of equations (2) (3), using the first angle of Fig. 1 and the fourth angle of Fig. 2. Then the curves in Fig. 4 are the images of the curves in Fig. 3 with reference to equation (1).

LINE OF REDUCTION.

21. *The general quadratic equation with real coefficients has a line of reduction. parallel to the y-axis.*

This line passes through the two critical points, or midway between them.

From what has been said concerning *order of images*, it can be easily shown

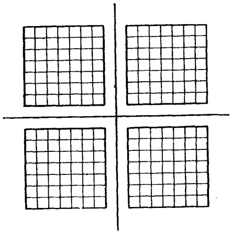


Fig. 1.

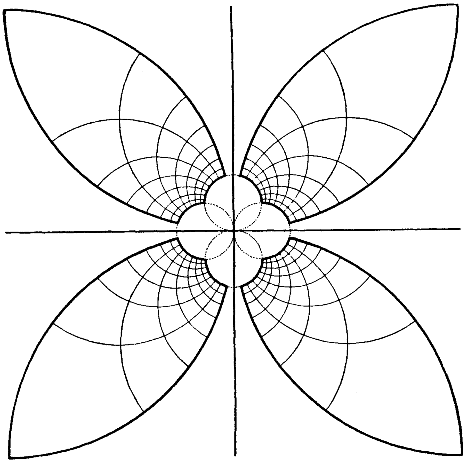


Fig. 2.

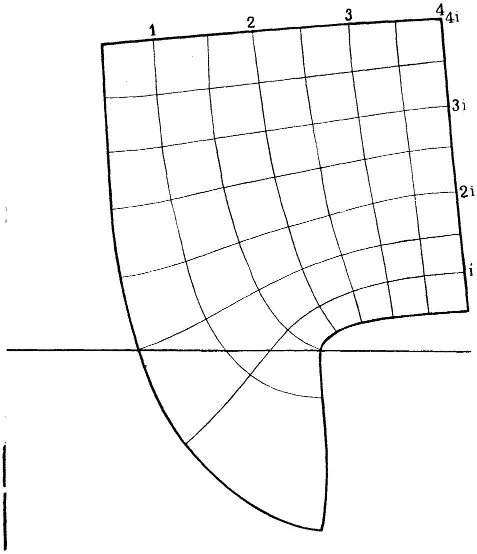


Fig. 3.

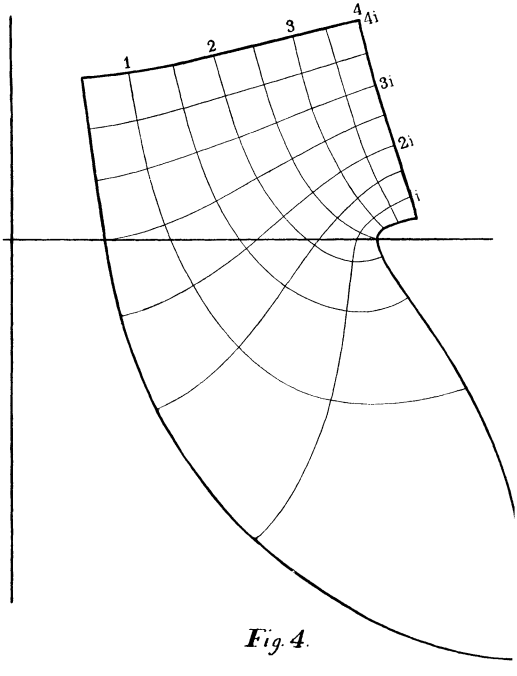


Fig. 4.

$I^{f_1}_{\Phi} = 0$ is, in general, of the fourth degree in X, Y. From $\mathfrak{p}\Phi = 0$, we have

$$y = -\frac{aXY + hYx + gY}{hX + bx + f} = -\frac{g}{h} Y$$

when $\frac{a}{h}(bx + f) = hx + g$; i. e., when

$$x = \frac{gh - fa}{ab - h^2}.$$

$$\mathfrak{r}\Phi = a(X^2 - Y^2) + b(x^2 - y^2) + 2h(Xx - Yy) + 2gX + 2fx + c = 0.$$

In this equation, substitute the above values for x and y. It will become the equation of

An hyperbola, if $ab > h^2$;
An ellipse, if $ab < h^2$;
A parabola, if $ab = h^2$.

If we eliminate u between Φ and $\frac{\partial\Phi}{\partial u}$, we have

$$z = \frac{gh - af}{ab - h^2} \pm \frac{\sqrt{\left|\begin{array}{c|ccc} -a & a, & h, & g \\ & h, & b, & f \\ & g, & f, & c \end{array}\right|}}{ab - h^2}.$$

The first part of the above expression is always real and represents the x-co-ordinate of a point midway between the two critical points. In case the second part is imaginary, this line passes through the points themselves. *Q. E. D.*

If the terms of the first degree be wanting, the line is the y-axis.

Φ OF THE FORM $\chi(u) - \psi(z)$.

22. Let us suppose χ of the mth degree and ψ of the nth. The order of I evidently depends upon the number of times that a_n can enter any term of the resultant as a factor. From (4) Art. 16, we see that it is np. But a_n is a function of the mth order in X, Y. Hence the order of I is mnp.

In the present case, the relation between u and z may be advantageously shown by a pair of systems representing the course of z and another pair representing that of u whose equations in either case shall be of the form

$$r_1 r_2 r_3 \ldots = c, \qquad \equiv F(r_1, r_2, r_3, \ldots), \qquad (1)$$

$$\theta_1 + \theta_2 + \theta_3 + \ldots = \alpha + 2k\pi, \equiv G(\theta_1, \theta_2, \theta_3, \ldots); \qquad (2)$$

wherein $r_1, r_2, \ldots, \theta_1, \theta_2, \ldots$ are the moduli and the arguments of the m or n factors into which $\chi(u)$ or $\psi(z)$ is resolvable.

It is easy to draw the normal at any point of (1), and the tangent at any point of (2).

For we have only to lay off on $r_1, r_2 \ldots$ distances proportional to

$$\frac{\partial F}{\partial r_1}, \ \frac{\partial F}{\partial r_2}, \ \ldots,$$

respectively, and, regarding them as forces, find the resultant. This gives the direction of the normal.

The direction of the tangent to (2) for a given point is found by taking distances on the r's proportional to

$$\frac{1}{r_1}\frac{\partial G}{\partial \theta_1}, \ \frac{1}{r_2}\frac{\partial G}{\partial \theta_2}, \ \ldots,$$

and by finding the resultant as before.

Since $$\frac{\partial F}{\partial r_1} = r_2 r_3 \ldots = \frac{c}{r_1},$$

and $$\frac{1}{r_1}\frac{\partial G}{\partial \theta_1} = \frac{1}{r_1}, \text{ etc.},$$

the two systems cut each other orthogonally, as is otherwise evident. For (1), (2) may represent either $\chi(u)$ or $\psi(z)$ as an auxiliary variable $s, = \chi(u) = \psi(z)$, describes a system of concentric circles and a system of straight lines through their centre. The s-pair of systems being orthogonal, so is the u-pair, or the z-pair, and therefore (1) and (2).

All asymptotes to (2) *pass through the centre of gravity of the zeros.*

Let the point of contact of the tangent be at infinity; then $r_1, r_2 \ldots$ become as $1 : 1 : \ldots$. Hence, we may regard them as representing equal parallel forces acting upon the zeros. But the resultant of such forces passes through the centre of gravity of the zeros where these points are regarded as material and all equal in mass.

For a real function, this centroid will lie on the x-axis.

When the points are three in number, their centre of gravity coincides with that of a plane triangle defined by them; when m in number, the co-ordinates are

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots}{m},$$

$$\bar{y} = \frac{y_1 + y_2 + y_3 + \ldots}{m}.$$

Equation (1), of course, approaches the equation of a circle, where the m asymptotes to (2) are radii dividing it into m compartments.

23. Resume the equation

$$\chi(u) = \psi(z) = s.$$

where s is an auxiliary variable. From what has just been said, it is obvious that when s describes a circle whose radius is infinity, the argument of u is increased by $2\pi/m$ and that of z by $2\pi/n$, for each revolution of s. Since s describes ${}^{m}_{n}$ circumferences while ${}^{u}_{z}$ describes one (and this is true for any path enclosing *all* the zeros) to every value of s correspond ${}^{m}_{n}$ distinct values of ${}^{u}_{z}$; and these m or n values form in the limit a circular system at infinity.

Now let s describe a very small curve about the origin; ${}^{u}_{z}$ will describe a similar curve about each of its ${}^{m}_{n}$ zeros. If, however, the zeros be not simple, the above statement holds true for circular paths only.

Φ OF THE FORM $u^m - \psi(z)$ or $\chi(u) - z^n$.

24. In this case, either the path, or the image, becomes a circle, which simplifies the representation.

In the case where χ or ψ is a quadratic function, $\chi(u) - \psi(z)$ can always by translation be put into the form

$$u'^2 = \psi(z) + c,$$

or

$$c' + \chi(u) = z'^2.$$

25. If we wish to avoid vectorial co-ordinates, we may suppose s to describe the two systems $x' = k$, $y' = k'$, where $s = x' + iy'$. Then z will describe the pair of systems whose equations are

$$\mathfrak{r}\psi = k, \quad \mathfrak{p}\psi = k';$$

and u,

$$\mathfrak{r}\chi = k, \quad \mathfrak{p}\chi = k'.$$

If one of the functions, say ψ, be of the form z^n, then the two systems described by z have for their equations

$$\frac{1}{\rho^n} = \frac{1}{k}\cos n\theta, \qquad (1)$$

$$\frac{1}{\rho^n} = \frac{1}{k'}\sin n\theta. \qquad (2)$$

These curves are inverses of the well-known curves

$$r^n = a^n \cos n\theta,$$

$$r^n = a'^n \sin n\theta.$$

The asymptotes to (1) and (2) pass through the origin, dividing the region about it into $2n$ equal angular compartments. The angles which the asymptotes to (1) make with the x-axis are $\pi/2n$, $3\pi/2n$, $5\pi/2n$, . . . ; the same for (2), 0, $2\pi/2n$, $4\pi/2n$, $6\pi/2n$, ...

Equation $\begin{Bmatrix}(1)\\(2)\end{Bmatrix}$ will give u hyperbolic branches for $\substack{+k\\+k'}$ and n conjugate ones for $\substack{-k\\-k'}$.

Φ OF THE THIRD DEGREE.

26. By means of the linear substitution,

$$u = \lambda u' + \mu z' + \nu,$$
$$z = \lambda' u' + \mu' z' + \nu',$$

we can always put $\Phi = 0$ into the form $\chi(u') = \psi(z')$. We have at our disposal four effective constants; therefore, after depriving Φ of the three terms containing u and z, we may impose one more condition. Let this be such a relation among the coefficients of χ or ψ that, by the addition of a constant to both sides of our equation, χ or ψ shall become a perfect cube. Then

$$(u' - \lambda)^3 = \psi(z') + k, \quad \text{or } u''^3 = \psi'(z').$$

Here, again, it will be advantageous for our pair of auxiliary systems to consist of 1° a system of concentric circles, and 2° their radii.

The equation $\chi(u') = \psi(z')$ *will become*

$$u'^3 = \psi'(z'),$$

by the above substitution, whenever the invariant S *of a ternary cubic vanishes.* For if

$$\begin{aligned}\Phi(u, z) \equiv\ & au^3 + bz^3 + c\nu^3 \\ & + 3a_2u^2z + 3a_3u^2\nu + 3b_1z^2u + 3b_3z^2\nu + 3c_1\nu^2u + 3c_2\nu^2z \\ & + 6muz\nu,\end{aligned}$$

then

$$a_2, a_3, b_1, c_1, m = 0,$$

which values substituted in S make it zero.

If Φ be homogeneous in u and z, we may separate the variables by the substitution,

$$u = \lambda u' + \mu z',$$
$$z = \lambda' u' + \mu' z'.$$

Φ OF THE FOURTH DEGREE.

27. *Whenever* Φ *is homogeneous in* u *and* z, *the variables may be separated by the substitution,*

$$u = \lambda u' + \mu z',$$
$$z = \lambda' u' + \mu' z',$$

provided that the invariant T *of a binary quartic vanishes.* For, let

$$\Phi \equiv au^4 + 4bu^3z + 6cu^2z^2 + 4duz^3 + ez^4;$$

then b, c, d must $= 0$.

$$\therefore T = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0.$$

In general, we cannot by the substitution,

$$u = \lambda u' + \mu z' + \nu,$$
$$z = \lambda' u' + \mu' z' + \nu',$$

separate u and z when Φ is of the fourth degree. For there are six terms which must disappear, and but four effective constants at our disposal in the matrix of transformation. Hence there exist two conditions among the coefficients whenever separation is possible. Let

$$\begin{aligned} \Phi \equiv\; & au^4 + bz^4 + c\nu^4 \\ & + 6fz^2\nu^2 + 6g\nu^2u^2 + 6hu^2z^2 + 12lu^2z\nu + 12mz^2\nu u + 12n\nu^2uz \\ & + 4a_2u^3z + 4a_3u^3\nu + 4b_1z^3u + 4b_3z^3\nu + 4c_1\nu^3u + 4c_2\nu^3z. \end{aligned}$$

To have the required form, obviously h, l, m, n, b_1, a_2 must each be zero. The sextinvariant of a ternary quartic is

$$\begin{vmatrix} a, & h, & g, & l, & a_3, & a_2 \\ h, & b, & f, & b_3, & m, & b_1 \\ g, & f, & c, & c_2, & c_1, & n \\ l, & b_3, & c_2, & f, & n, & m \\ a_3, & m, & c_1, & n, & g, & l \\ a_2, & b_1, & n, & m, & l, & h \end{vmatrix}.$$

But the last row becomes zero. Hence, *whenever u and z can be separated by the above substitution, the sextinvariant vanishes.*

The converse of this statement is not in general true, because there is, as we have seen, one more condition to be satisfied. This may not be as simple as the one just found.

Of course we can deprive Φ of the six terms whose coefficients are a_2, b_1, etc., by means of a homographic substitution; but that would be less convenient for the present purpose.

GENERAL CASE, Φ OF THE FORM $\omega(u, z) + \psi(z)$ OR $\chi(u) + \omega(u, z)$.

28. If ω be of a degree not higher than the fourth, (in u for the first case, in z for the second), it is possible to separate u and z by introducing irrational expressions. But owing to the complicated results, the degree of ω is practically limited to the second in the general case. There are a few special cases where separation is possible; e. g.

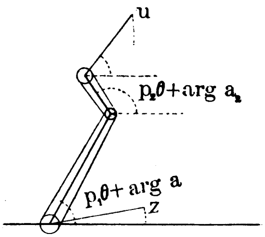
$$u^{2m} + \xi u^m = \zeta,$$

where ξ, ζ are any functions of z; and so for $z^{2n}+$, etc.

29. The curves $I_{\phi(z)}^{(\rho=k)}$ admit of a simple organic description when $\varphi(z)$ is of the form

$$\Sigma a z^{p},$$

where a is general, and p is any real number, entire or fractional.



Imagine a series of bars whose lengths are ρ^{p_1} mod. a_1, ρ^{p_2} mod. a_2, . . . , and suppose them jointed together and moved with angular velocities proportional to p_1, p_2, . . . , starting with angles of inclination to the x-axis equal to arg. a_1, arg. a_2, ...

The constant term of φ may be regarded as a bar without motion.

It does not seem practicable to describe the orthogonal set, $I_{\phi(z)}^{(\theta=a)}$, by any similar mechanism.

APPROXIMATE EQUATIONS OF IMAGES.

30. Let $u=\varphi(z)$, where φ is any real function, algebraic or transcendental. Now if z move in a region where $\mathfrak{r}\varphi$ and $\mathfrak{p}\varphi$ are convergent, we may obtain approximate expressions for $I_{\phi(z)}^{(x=a)}$ or $I_{\phi(z)}^{(y=\beta)}$ *whose degree shall be equal to the highest power of the y or x used in the development.*

Let $x=a$; then we have

$$X = a_0 + a_2 y^2 + a_4 y^4 + \ . \ . \ . \ ,$$
$$Y = a_1' y + a_3' y^3 + a_5' y^5 + \ . \ . \ . \ ;$$

$$\therefore\ I_{\phi_2}^{(x=a)} = \begin{vmatrix} a_2, & 0, & a_0 - X \\ a_1',. & -Y, & 0 \\ 0, & a_1', & -Y \end{vmatrix} = 0.$$

The subscript of φ indicates the highest power of y taken.

$$I_{\phi_3}^{(x=a)} = \begin{vmatrix} a_2, & 0, & a_0 - X, & 0, & 0 \\ 0, & a_2, & 0, & a_0 - X, & 0 \\ 0, & 0, & a_2, & 0, & a_0 - X \\ a_3', & 0, & a_1', & -Y, & 0 \\ 0, & a_3', & 0, & a_1', & -Y \end{vmatrix} = 0.$$

The general equation takes one of the two forms:

(1.) *n odd*

$$I_{\phi n}^{x=a} = X_0 Y^{n-1} + X_2 Y^{n-3} + X_4 Y^{n-5} + \ . \ . \ . \ + X_{n-2} Y^2 + X_n Y^0 = 0,$$

(2.) *n even*

$$I_{\phi_n}^{x=a} = X_0 Y^n + X_1 Y^{n-2} + X_3 Y^{n-4} + \; . \; . \; . + X_{n-3} Y^2 + X_{n-1} Y^0 = 0,$$

where X_r is a polynomial of the rth degree in X.

Since $x = a$ is a path symmetric with reference to the x-axis, it follows from Art. 13 that I is symmetric with reference to the X-axis. In other words, Y can enter by even powers only.

In the other system $y = \beta$, we develop X and Y in powers of x; we then have

$$b_0 + b_2 x^2 + b_4 x^4 + \; . \; . \; . \underset{\phi \text{ even}}{= X,} \quad \underset{\phi \text{ odd}}{= Y,}$$

$$b_1' x + b_3' x^3 + b_5' x^5 + \; . \; . \; . \underset{\phi \text{ even}}{= Y,} \quad \underset{\phi \text{ odd}}{= X.}$$

When φ is an even function, $I_{\phi}^{(y=\beta)}$ is of the same form as $I_{\phi}^{x=a}$; if an odd function, it will be of the same form after interchanging X and Y.

LINES OF EQUAL EXPANSION AND THEIR ORTHOGONAL TRAJECTORIES.

31. If each element along the u-path bears a constant ratio to the corresponding element of the z-path, this latter is called a *line of equal expansion*.

Let dA, da represent elementary u- and z-areas, and let accents denote differentiation. Then

$$dA = (\text{mod. } du)^2, \quad da = (\text{mod. } dz)^2.$$

Let $x = a$ or $y = \beta$; then mod. $dz = \surd[(dx)^2 + (dy)^2]$ becomes dy or dx, mod. du becomes $\surd(X'^2 + X_{\prime}^2) \;\; dy, dx$;

$$\therefore \frac{dA}{da} = X'^2 + X_{\prime}^2 = k,$$

is the equation of the system of lines of equal expansion where k is the parameter.

The system $\dfrac{X'}{X_{\prime}} = c$ *cuts the system* $X'^2 + X_{\prime}^2 = k$ *orthogonally.*

For, differentiating the former, we have

$$\frac{dy}{dx} = -\frac{X''X_{\prime} - X'X_{\prime}'}{X_{\prime}'X_{\prime} - X'X_{\prime\prime}} = \frac{X'X_{\prime}' + X_{\prime}X_{\prime\prime}}{X'X'' + X_{\prime}X_{\prime}'}, \tag{1}$$

since $X'' = -X_{\prime\prime}$. The latter system gives

$$\frac{dy}{dx} = -\frac{X'X'' + X_{\prime}X_{\prime}'}{X'X_{\prime}' + X_{\prime}X_{\prime\prime}} \tag{2}$$

But (1) is the reciprocal of (2) with its sign changed.

FOCI.

32. A *focus* of a function u is a point in the u-plane in whose immediate vicinity the transformed elements are infinitely small in comparison with their for-

mer magnitude. The corresponding point in the z-plane may be called an *anti-focus.*

To find the anti-foci of a given function.

We have, by definition,

$$\frac{\text{mod.}\,du}{\text{mod.}\,dz} = 0.$$

$$\therefore \sqrt{(X'^2 + Y'^2)} = 0; \text{ i. e. } X' = i\,Y', = -\,i X_{,}.$$

This equation can be true only when X' and $X_{,}$ are separately equal to zero, since these functions are by hypothesis both real:

$$\therefore \begin{cases} X' = 0, \\ X_{,} = 0, \end{cases}$$

determine the anti-foci.

TRANSCENDENTAL FUNCTIONS.

33. When $u = e^z$,

then
$$X = e^x \cos y,$$
$$Y = e^x \sin y.$$

If $x = \alpha$, we have $X^2 + Y^2 = e^{2\alpha}$; i. e. u describes a circle whose centre is at the origin and whose radius is equal to e^α.

If $y = \beta$, then $Y = X \tan \beta$; i. e. u describes a straight line through the origin making an angle β with the X-axis.

If $f(x,y) = 0$ be the equation of the path, then that of the image is

$$f\left[\tfrac{1}{2}\log\,(X^2 + Y^2), \tan^{-1}\frac{Y}{X}\right] = 0.$$

34. When $u = \sin z$,

$$X = \sin x \cosh y,$$
$$Y = \cos x \sinh y.$$

Let $x = \alpha$, and let $a \equiv \cos \alpha$, $b \equiv \sin \alpha$;

then
$$\frac{X^2}{b^2} - \frac{Y^2}{a^2} = 1.$$

Next, let $y = \beta$, and let $a' \equiv \cosh \beta$, $b' \equiv \sinh \beta$;

then
$$\frac{X^2}{a'^2} + \frac{Y^2}{b'^2} = 1.$$

When $u = \cos z$,

$$X = \cos x \cosh y,$$
$$Y = -\sin x \sinh y;$$
$$\therefore \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1,$$
$$\frac{X^2}{a'^2} + \frac{Y^2}{b'^2} = 1.$$

The ellipses are the same as before; but the hyperbolas have their axes interchanged. All the ellipses and hyperbolas are bi-confocal.

35. $u = \text{sn}\, z$.

(1.) First of all, let us examine the region about the focus $X = 1$, $Y = 0$ whose anti-focus is $x = K$, $y = 0$.

Let $z = K + z'$; then

$$u = \text{sn}\,(K + z') = \frac{\text{cn}\, z'}{\text{dn}\, z'} = \frac{1 - \frac{1}{2}z'^2 + \ .\ .\ .}{1 - \frac{1}{2}z'^2 . k^2 + \ .\ .\ .} = 1 - \frac{1-k^2}{2} z'^2 + \ .\ .\ .\ .$$

Neglecting powers of z' beyond the second, we have

$$X = 1 - \tfrac{1}{2}(1 - k^2)(x'^2 - y'^2).$$
$$Y = -(1 - k^2)\, x' y'.$$

Let $x' = \alpha$, and write X' for $X - 1$; then

$$-X' = \tfrac{1}{2}(1 - k^2)\left[\alpha^2 - \frac{Y^2}{(1 - k^2)^2 \alpha^2}\right]$$

is the equation of $I^{(x'=\alpha)}_{\text{sn}\, z}$, where the focus is the origin. For the system $y = \beta$, we have

$$-X' = \tfrac{1}{2}(1 - k^2)\left[\frac{Y^2}{(1 - k^2)\beta^2} - \beta^2\right].$$

In like manner, may be discussed the region about the focus $u = 1/k$ whose anti-focus is $x = K$, $y = K'$. In this case,

$$\text{sn}\,(K + K'i + z') = \frac{1}{k}\frac{\text{dn}\, z'}{\text{cn}\, z'} = \frac{1}{k}\left(1 + \frac{1 - k^2}{2} z'^2 + \ .\ .\ .\right).$$

If z be near $K'i$, and u near ∞, let $z = K'i + z'$; then

$$u = \text{sn}\,(K'i + z') = \frac{1}{k\, \text{sn}\, z'} \doteq \frac{1}{kz'};$$

z' being very small.

If z' describe the system $x' = -\alpha, u$ will describe a system of circles whose centres lie upon the X-axis and which have a common point of contact at the origin. The radius of any particular circle will be equal to $1/2k\alpha$.

Now let $y' = -\beta$ in the equation

$$u = \frac{1}{kz'}.$$

These circles have their centres upon the Y-axis at distances equal to $1/2k\beta$ from the origin, where they have a common point of contact.

(2.) Curve of maximum abscissas.

$$u = \operatorname{sn}(x + iy) = \frac{\operatorname{sn} x \operatorname{cn} iy \operatorname{dn} iy + \operatorname{sn} iy \operatorname{cn} x \operatorname{dn} x}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy},$$

$$X = \frac{\operatorname{sn} x \operatorname{cn} iy \operatorname{dn} iy}{1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy},$$

$$\frac{\partial X}{\partial x} = \frac{\operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy\,(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy) + 2k^2 \operatorname{sn}^2 x \operatorname{cn} x \operatorname{dn} x \operatorname{sn}^2 iy \operatorname{cn} iy \operatorname{dn} iy}{(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy)^2}$$

$$= \frac{1 + k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy}{(1 - k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy)^2} \operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy.$$

$\partial X/\partial x$ may be made equal to zero by equating to zero, either

$$\operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy, \quad \text{or} \quad 1 + k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy$$

y, the parameter, being less than K'.

If $x = K, \quad \operatorname{cn} x \operatorname{dn} x \operatorname{cn} iy \operatorname{dn} iy = 0;$

therefore that part of the X-axis which joins the foci 1 and $1/k$ is the locus of the points of contact of a tangent, parallel to the Y-axis, with the system

$$I^{(y=\beta)}_{\operatorname{sn} z}.$$

The other locus of vertical tangents is determined by aid of the equation

$$1 + k^2 \operatorname{sn}^2 x \operatorname{sn}^2 iy = 0, \quad \text{or} \quad \operatorname{sn} x \operatorname{sn} iy = \pm i/k.$$

It was for this locus that the above name was intended. Obviously the curve of maximum abscissas for $I^{(y=\beta)}_{\operatorname{sn} z}$ is the curve of maximum ordinates for $I^{(x=\alpha)}_{\operatorname{sn} z}$.

(3.) EXAMPLE. Suppose $k = \frac{1}{2}$; and let the region of the z-plane considered, be a rectangle whose edges are K and K' in length.

For this value of k, $K = 1.68575$ and $K' = 2.1565$. By means of expansions and addition formulae, we construct the following tables:—

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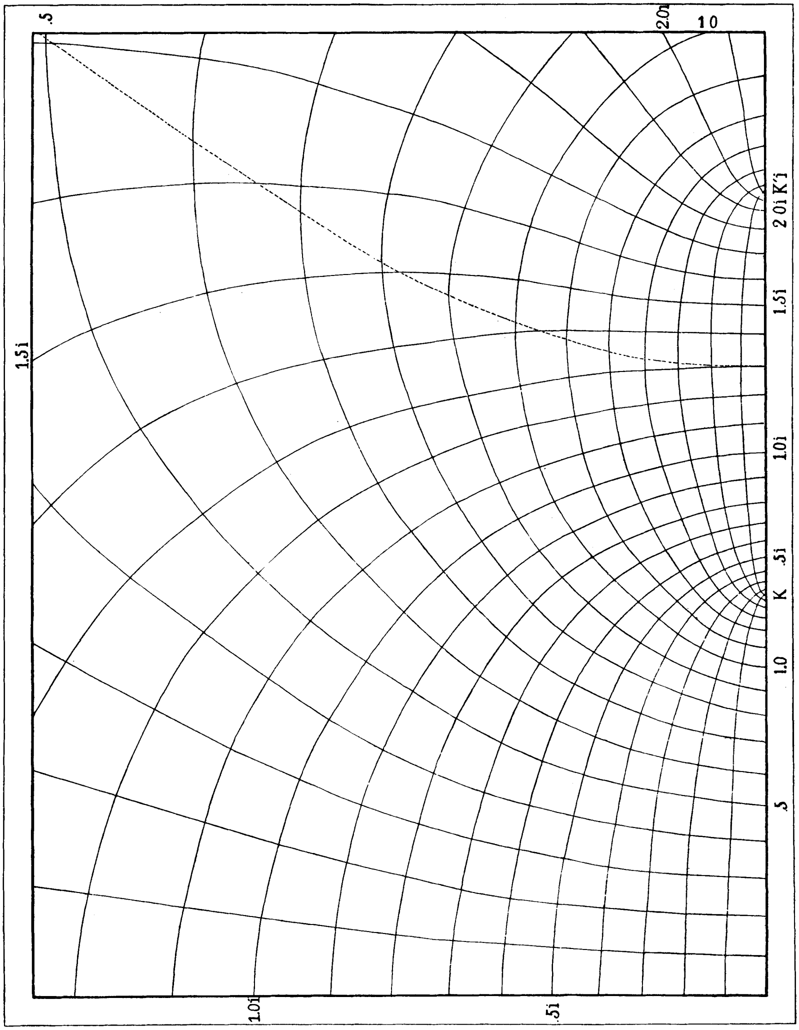

z	sn z	cn z	dn z.	z	$-i$ sn iz	cn iz	dn iz
.1	.09979	.99501	.99875	.1	.10021	1.00501	1.00125
.2	.19835	.98013	.99507	.2	.20168	1.02013	1.00507
.3	.29447	.95566	.98910	.3	.30572	1.04569	1.01162
.4	.38704	.92206	.98110	.4	.41374	1.08221	1.02117
.5	.47508	.87994	.97138	.5	.52730	1.13051	1.03417
.6	.55773	.83002	.96033	.6	.64822	1.19171	1.05121
.7	.63429	.77310	.94839	.7	.77864	1.26739	1.07311
.8	.70421	.70999	.93596	.8	.92121	1.35964	1.10098
.9	.76708	.64155	.92352	.9	1.07929	1.47135	1.13632
1.0	.82263	.56857	.91149	1.0	1.25728	1.60647	1.18118
1.1	.87069	.49183	.90026				
1.2	.91117	.41203	.89019	1.5	2.77546	2.95011	1.71050
1.3	.94405	.32981	.88159				
1.4	.96934	.24572	.87470	2.0	12.7131	12.7523	6.43471
1.5	.98706	.16039	.86973				
1.6	.99724	.07422	.86682				
K	1.00000	.00000	.86603	K'	∞	∞	∞

Suppose a sufficient number of real and purely imaginary values of z to have been taken, and the values of sn z, cn z, and dn z computed and thus tabulated; then, by the addition formulae, we can find any mixed value of

$$u \; (= \mathrm{sn}\, z,\ \mathrm{cn}\, z,\ \text{or}\ \mathrm{dn}\, z.)$$

The dotted line upon the chart is the curve of maximum abscissas for the system $I^{(y=\beta}_{\mathrm{sn}\, z}$. To find where this curve intersects the X-axis, make $x = K$; $\therefore$ sn $x = 1$; sn x sn $iy = 2i$ becomes $-i$ sn $iy = 2$. Had we a complete table, we should find y equal to about 1.3. Therefore this curve meets the X-axis near the point where the curve $I^{(y=1.3)}_{\mathrm{sn}\, z}$ meets it.

About $\frac{1}{6}$ of the part of the z-plane considered, lies, when transformed, beyond the limits of our chart.

By making use of the same computations we could, of course, construct charts for cn z, dn z, $\mathrm{sn}^{-1} z$, etc.

SOLID IMAGES.

36. Let the right versor i operate in planes parallel to xy, and j, in planes parallel to xz, then the affix of any point in space is

$$u = x + iy + jz.$$

Let
$$U = \varphi(u) = X + iY + jZ.$$

By Taylor's Theorem,

$$U = \varphi(x) + \frac{iy + jz}{1}\frac{\partial\varphi(x)}{\partial x} + \frac{(iy+jz)^2}{2!}\frac{\partial^2\varphi(x)}{\partial x^2} + \ldots;$$

whence, expanding the binomials $(iy + jz)^r$, and omitting all terms of the form $Ki^m j^n y^m z^n$ where m and n are both odd, we have

$$\begin{aligned} X = \varphi(x) - \frac{y^2+z^2}{2!}\varphi''(x) + \frac{y^4 + 6y^2z^2 + z^4}{4!}\varphi^{\text{iv}}(x) \\ - \frac{y^6 + 15y^4z^2 + 15y^2z^4 + z^6}{6!}\varphi^{\text{vi}} + \ldots, \end{aligned} \tag{1}$$

$$Y = y\varphi'(x) - \frac{y^3 + 3yz^2}{3!}\varphi'''(x) + \frac{y^5 + 10y^3z^2 + 5yz^4}{5!}\varphi^{\text{v}}(x) - \ldots, \tag{2}$$

$$Z = z\varphi'(x) - \frac{z^3 + 3zy^2}{3!}\varphi'''(x) + \frac{z^5 + 10z^3y^2 + 5zy^4}{5!}\varphi^{\text{v}}(x) - \ldots. \tag{3}$$

It is easily shown, from (1), (2), (3), when z is constant, that

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y},$$

$$\frac{\partial Y}{\partial x} = -\frac{\partial X}{\partial y};$$

and when y is constant, that

$$\frac{\partial X}{\partial x} = \frac{\partial Z}{\partial z},$$

$$\frac{\partial Z}{\partial x} = -\frac{\partial X}{\partial z}.$$

Let us return for a moment to the consideration of plane images.

(a) If $u = x + iy + jz$ move along any two intersecting curves whose plane is parallel to xy, then the projections of their images upon XY intersect at the same angles as did the original paths of the variable u. For we have already seen that if

$$U = X + iY,$$

and if z be constant, U is a function of $u' = x + iy$. In like manner, if

$$U = X + jZ,$$

and if y be constant, U is a function of $u'' = x + jz$. Therefore, in either case, all angles are preserved in the projections (upon the plane XY or XZ).

Let

$$X = \varphi_1 (x, y, z), \tag{1'}$$

$$Y = \varphi_2 (x, y, z), \tag{2'}$$

$$Z = \varphi_3 (x, y, z); \tag{3'}$$

also, let the equations of the path of u be $f_1 (x, y, z) = 0$, $f_2 (x, y, z) = 0$. If we eliminate x, y, z, the two resulting equations in X, Y, Z determine a curve in space of three dimensions—the path of U.

We shall next discuss the infinitesimal elements of the U-space.

If V be any function ψ of x, y, z, and $V_1 = \psi (x_1, y_1, z_1)$, $V_2 = \psi (x_2, y_2, z_2)$ be two special values where x_1, y_1, z_1; x_2, y_2, z_2 are points very near together; then, if x_1, y_1, z_1 become $x_1 + h$, $y_1 + k$, $z_1 + l$, and if x_2, y_2, z_2 become $x_2 + h$, $y_2 + k$, $z_2 + l$, respectively, V_1 and V_2 are increased by equal amounts as $h, k, l \doteq 0$. For

$$V_1 + H_1 = \psi (x_1 + h,\ y_1 + k,\ z_1 + l) = V_1 + h \frac{\partial V_1}{\partial x_1} + k \frac{\partial V_1}{\partial y_1} + l \frac{\partial V_1}{\partial z_1} + \ldots,$$

$$V_2 + H_2 = \psi (x_2 + h,\ y_2 + k,\ z_2 + l) = V_2 + h \frac{\partial V_2}{\partial x_2} + k \frac{\partial V_2}{\partial y_2} + l \frac{\partial V_2}{\partial z_2} + \ldots.$$

Now $H_1 \doteq H_2$; for, as $x_1 \doteq x_2$, $y_1 \doteq y_2$, $z_1 \doteq z_2$;

$$\frac{\partial V_1}{\partial x_1} - \frac{\partial V_2}{\partial x_2} = \varepsilon,$$

where ε is an infinitesimal of the first order;

$$\therefore h \left\{ \frac{\partial V_1}{\partial x_1} - \frac{\partial V_2}{\partial x_2} \right\} = h\varepsilon,$$

an infinitesimal of the second order. So proceed with the quantities multiplied by k and l;

$$\therefore H_1 - H_2 \doteq 0.$$

Let x_1, y_1, z_1; x_2, y_2, z_2 be the co-ordinates of the extremities of a very short; and we may suppose, straight line; then, of course, $x_1 + h$, $y_1 + k$, $z_1 + l$, $x_2 + h$, $y_2 + k$, $z_2 + l$ determine an equal and parallel line. Suppose

$$X_1 = \psi_1 (x_1, y_1, z_1),$$

$$Y_1 = \psi_2 (x_1, y_1, z_1),$$

$$Z_1 = \psi_3 (x_1, y_1, z_1),$$

$$X_2 = \psi_1 (x_2, y_2, z_2),$$

$$\ldots \qquad \ldots$$

to be co-ordinates of the extremities of a very short line; then, if the line x_1, y_1, z_1; x_2, y_2, z_2 move parallel to itself, so does the line

$$X_1,\ Y_1,\ Z_1;\ X_2,\ Y_2,\ Z_2.$$

For, giving to x, y, z the necessary increment h, k, l, we have

$$\begin{aligned} &X_1 + H \text{ for } X_1, \\ &X_2 + H \text{ `` } X_2, \\ &Y_1 + K \text{ `` } Y_1, \\ &\ldots\quad\ldots \end{aligned}$$

(*b*) Hence, if the u-space be divided into elementary parallelopipeds, so is the U-space, regardless of the nature of the function ψ.

Therefore, if the u-space be divided into elementary cubes by planes parallel to the co-ordinate planes, the transformed elements will be parallelopipeds, the projections of the XY and XZ faces of which, upon the co-ordinate planes XY, XZ, respectively, will be perfect squares. (*a*), (*b*).

EXAMPLES.

(1) If

$$\begin{aligned} U &= u^2, \\ X &= x^2 - y^2 - z^2, \\ Y &= 2xy, \\ Z &= 2xz. \end{aligned}$$

(2) If

$$\begin{aligned} U &= e^u, \\ X &= e^x \cos y \cos z, \\ Y &= e^x \sin y \cos z, \\ Z &= e^x \cos y \sin z. \end{aligned}$$

(3) If

$$\begin{aligned} U &= \sin u, \\ X &= \sin x \cosh y \cosh z, \\ Y &= \cos x \sinh y \cosh z, \\ Z &= \cos x \cosh y \sinh z. \end{aligned}$$

These are analogous to plane images, and are reduced to them by making $z = 0$.